2

TENSOR ALGEBRA

Introduction : The concept of a tensor has its origin in the developments of differential geometry by Gauss, Riemann and Christoffel. The emergence of tensor calculus, as a systematic branch of mathematics is due to Ricci and his pupil Levi-Cita.

During the study of elasticity and physics of crystals, the physicists came across with new kind of quantities called afterwards tensors, more complex than vectors. During the study of stresses and tensions in the interior of a deformed body, they discover a collection of six numbers inseparable from one another, which behave like the six components of a certain quantity.

Def. Tensor. The physical quantities which have more than one direction are represented by the mathematical entities, are called tensors.

Scalars and vectors are special cases of tensors.

2.0. CONTRAVARIANT AND CO-VARIANT VECTORS

(Kanpur Riemannian Geom. 1975; Banaras Riemannian Geom. 1970)

Let $A^i, i = 1, 2, \ldots, n$, be n functions of co-ordinates $x^1, x^2, \ldots x^n$. If the quantities A^i are transformed to another co-ordinate system $x'^1, x'^2, x'^3, \ldots x'^n$ according to the rule $A'^i = A^\alpha \dfrac{\partial x'^i}{\partial x^\alpha}$,

then the functions A^i are called components of **contravariant vector.**

Let $A_i, i = 1, 2, 3, \ldots, n$, be n functions of co-ordinates $x^1, x^2, \ldots x^n$. If the quantities A_i are transformed to another co-ordinate system $x'^1, x'^2, \ldots, x'^n$ according to the rule

$$A_i' = A_\alpha \frac{\partial x^\alpha}{\partial x'^i},$$

then the functions A_i are called components of a **covariant vector.**

The contravariant (or covariant) vector is also called a **contravariant** (or **covariant) tensor** of rank one.

2.1. TENSOR

(Kanpur Riemannian Geom. 1985, 69; Banaras Riemannian Geom. 62)

Let A^{ij} $(i, j = 1, 2, 3, \ldots, n)$ be n^2 functions of co-ordinates $x^1, x^2, \ldots, x^n$ and let these transform to A'^{ij} in another co-ordinate system $x'^1, x'^2, \ldots, x'^n$ according to the rule

$$A'^{ij} = A^{\alpha\beta} \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x'^j}{\partial x^\beta}.$$

Then A^{ij} are called components of a **contravariant tensor of rank two.**

Similarly if A_{ij} $(i, j = 1, 2, \ldots, n)$ be n^2 functions of co-ordinates $x^1, x^2, \ldots x^n$ and if A_{ij} are transformed to A'_{ij} in another co-ordinate system $x'^1, x'^2, \ldots, x'^n$ by the rule

$$A'_{ij} = A_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^i} \frac{\partial x^\beta}{\partial x'^j},$$

then A_{ij} are said to be **covariant tensor of rank two.**

Finally, if the n^2 functions A_j^i $(i, j = 1, 2, \ldots, n)$ of co-ordinates x^i be transformed of $A_j'^i$ in another co-ordinate system x'^i according to the rule

$$A'^i_j = A^\alpha_\beta \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^j}$$

then A_j^i are said to be components of **mixed tensor of rank two.**

The upper position of the suffix is reserved to indicate contravariant character whereas the lower position of the suffix is reserved to indicate covariant character.

The **rank** of a tensor is defined as the total number of real indices per component.

A tensor of the type $A^{i_1 i_2 \ldots i_r}_{j_1 j_2 \ldots j_s}$ is known as a tensor of the type (r, s). Thus the tensors A_{ij} and A^{ij} are of the type $(0, 2)$ and $(2, 0)$ respectively.

(Kanpur Riemann Geom. 1987, 85)

Instead of saying that "let A_{ij} be the components of second rank covariant tensor", we always say, "let A_{ij} be second rank covariant tensor." Tensors of higher ranks are defined as follows :

(i) Covariant tensor of rank l :

$$A'_{i_1 i_2 \ldots i_l} = A_{\alpha_1 \alpha_2 \ldots \alpha_l} \frac{\partial x^{\alpha_1}}{\partial x'^{i_1}} \frac{\partial x^{\alpha_2}}{\partial x'^{i_2}} \cdots \frac{\partial x^{\alpha_l}}{\partial x'^{i_l}}$$

(ii) Contravariant tensor of rank l :

$$A'^{i_1 i_2 \ldots i_l} = A^{\alpha_1 \alpha_2 \ldots \alpha_l} \frac{\partial x'^{i_1}}{\partial x^{\alpha_1}} \frac{\partial x'^{i_2}}{\partial x^{\alpha_2}} \cdots \frac{\partial \alpha'^{i_l}}{\partial x^{\alpha_l}}$$

(iii) Mixed tensor of rank $l + m$:

$$A'^{i_1 i_2 \ldots i_l}_{j_1 j_2 \ldots j_m} = A^{\alpha_1 \alpha_2 \ldots \alpha_l}_{\beta_1 \beta_2 \ldots \beta_m} \frac{\partial x'^{i_1}}{\partial x^{\alpha_1}} \frac{\partial x'^{i_2}}{\partial x^{\alpha_2}} \cdots \frac{\partial x'^{i_l}}{\partial x^{\alpha_l}} \frac{\partial x^{\beta_1}}{\partial x'^{j_1}} \frac{\partial x^{\beta_2}}{\partial x'^{j_2}} \cdots \frac{\partial x^{\beta_m}}{\partial x'^{j_m}}$$

A tensor of rank m in n dimensions has n^m components.

Thus the general form of tensor includes vectors (tensors of rank one) and scalars (tensors of rank zero).

The key property of a tensor is the transformation law of its components. The precise form of this transformation law is a consequence of the physical or geometric meaning of the tensor.

Problem. ***How many components does a tensor of rank 3 have in a space of 4 dimensions?*** *(Kanpur B. Sc. 1993)*

Solution. Total number of components $= n^m = 4^3 = 64$.

where n = dimension $= 4$, m = rank $= 3$.

2.2. (a) GRADIENT

Let ϕ be a scalar function of co-ordinates $x^1, x^2, ..., x^n$. The gradient ϕ, denoted by grad ϕ or $\nabla\phi$, is defined as its ordinary partial derivative. Thus

$$\nabla\phi = \frac{\partial \phi}{\partial x^i}.$$

2.2. (b) TENSOR FIELD. DEFINITION

If a tensor is defined at all points of a curve or throughout the space V_n itself, then we say that it forms a tensor field. However we will use interchangeably the terms tensor and tensor-field.

Theorem 1. ***To show that there is no distinction between contravariant and covariant vectors when we restrict ourselves to rectangular Cartesian transformations of co-ordinates.***

(Vikram University Ujjain 1995; Banaras Riemannian Geometry 1963)

Proof. Let (x, y) be co-ordinates of a point P w.r.t. orthogonal cartesian axes X and Y. Let (x', y') be the co-ordinates of the same point P relative to orthogonal cartesian axes X' and Y'. Let (l_1, m_1) and (l_2, m_2) be direction cosines of the axes-X' and Y' respectively. Then we have the relation

$$\left.\begin{aligned} x' &= l_1 x + m_1 y \\ y' &= l_2 x + m_2 y \end{aligned}\right\} \quad \text{... (1)}$$

From which we have

$$\left.\begin{aligned} x &= l_1 x' + l_2 y' \\ y &= m_1 x' + m_2 y' \end{aligned}\right\} \quad \text{... (2)}$$

Let $x^1 = x, x^2 = y.$

Consider contravariant transformation

$$A'^i = A^a \frac{\partial x'^i}{\partial x^a} = A^1 \frac{\partial x'^i}{\partial x^1} + A^2 \frac{\partial x'^i}{\partial x^2}.$$

$\therefore$
$$A'^1 = A^1 \frac{\partial x'^1}{\partial x^1} + A^2 \frac{\partial x'^1}{\partial x^2}$$

and
$$A'^2 = A^1 \frac{\partial x'^2}{\partial x^1} + A^2 \frac{\partial x'^2}{\partial x^2}$$

i.e.,
$$A'^1 = A^1 \frac{\partial x'}{\partial x} + A^2 \frac{\partial x'}{\partial y}$$

and
$$A'^2 = A^1 \frac{\partial y'}{\partial x} + A^2 \frac{\partial y'}{\partial y}.$$

Writing these with the help of (1),

$$\left.\begin{aligned} A'^1 &= A^1 l_1 + A^2 m_1 \\ A'^2 &= A^1 l_2 + A^2 m_2 \end{aligned}\right\} \quad \text{... (3)}$$

Consider co-variant transformation

$$A_i' = A_\alpha \frac{\partial x^\alpha}{\partial x'^i} = A_1 \frac{\partial x^1}{\partial x'^i} + A_2 \frac{\partial x^2}{\partial x'^i}$$

$\therefore$
$$A'_1 = A_1 \frac{\partial x^1}{\partial x'^1} + A_2 \frac{\partial x^2}{\partial x'^1}$$

and
$$A'_2 = A_1 \frac{\partial x^1}{\partial x'^2} + A_2 \frac{\partial x^2}{\partial x'^2}$$

i.e.,
$$A'_1 = A_1 \frac{\partial x}{\partial x'} + A_2 \frac{\partial y}{\partial x'}$$

$$A'_2 = A_1 \frac{\partial x}{\partial y'} + A_2 \frac{\partial y}{\partial y'}$$

Writing these with the help of (2).

$$\left.\begin{aligned} A'_1 &= A_1 l_1 + A_2 m_1 \\ A'_2 &= A_1 l_2 + A_2 m_2 \end{aligned}\right\} \quad \text{... (4)}$$

Comparing (3) and (4), the required result follows.

Theorem 2. ***(a) To prove that the transormations of a contravariant vector form a group.***

(Kanpur Riemannian Geom. 2000, 1997, Gujrat 1972)

Or, To prove that the transformations of a contravariant vector is transitive.

(Kanpur B. Sc. 1998, Meerut 1993)

Proof. Let A^i be a contravariant vector. Consider co-ordinate transformations : $x'^i = x'^i (x^k)$, $x''^i = x''^i (x'^k)$, that is,

$$\begin{array}{ccccc} x^i & \longrightarrow & x'^i & \longrightarrow & x''^i \\ (i) & \longrightarrow & (ii) & \longrightarrow & (iii) \\ A^i & \longrightarrow & A'^i & \longrightarrow & A''^i \end{array}$$

In case of $x^i \longrightarrow x'^i$, we have

$$A'^\alpha = A^p \frac{\partial x'^\alpha}{\partial x^p} \quad \text{... (1)}$$

In case of transformation $x'^i \longrightarrow x''^i$,

$$A''^i = A'^\alpha \frac{\partial x''^i}{\partial x'^\alpha} = A^p \frac{\partial x'^\alpha}{\partial x^p} \frac{\partial x''^i}{\partial x'^\alpha}, \text{ by (1)}$$

or,
$$A''^i = A^p \frac{\partial x''^i}{\partial x^p}$$

This proves that if we make the transformation from (i) to (iii), we get the same law of transformation. This property is expressed by saying that transformations of a contravariant vector form a group.

Theorem 2. ***(b) To prove that the transformations of a covariant vector is transitive.***

Proof. Let A_i be a covariant vector. Consider the coordinate transformation:

$$\begin{array}{ccccc} x^i & \longrightarrow & x'^i & \longrightarrow & x''^i \\ (i) & \longrightarrow & (ii) & \longrightarrow & (iii) \\ A_i & \longrightarrow & A'_i & \longrightarrow & A''_i \end{array}$$

In case of $x^i \longrightarrow x'^i$, we have

$$A'_\alpha = A_p \frac{\partial x^p}{\partial x'^\alpha} \quad \text{... (1)}$$

In case of $x'^i \longrightarrow x''^i$,

$$A_i'' = A_\alpha' \frac{\partial x'^\alpha}{\partial x''^i} \quad \text{... (2)}$$

Combining (1) and (2), we get

$$A_i'' = \left(A_p \frac{\partial x^p}{\partial x'^\alpha} \right) \frac{\partial x'^\alpha}{\partial x''^i}$$

$$= A_p \left(\frac{\partial x^p}{\partial x'^\alpha} \frac{\partial x'^\alpha}{\partial x''^i} \right)$$

or,

$$A_i'' = A_p \frac{\partial x^p}{\partial x''^i}$$

This proves that if we make direct transformation from (i) to (iii), we get the same law of transformation. This property is expressed by saying that transformations of a covariant vector is transitive.

Theorem 2. (c) ***To prove that the equations of transformation of a mixed tensor (tensor) possess the group property. (or transitive property).***

(Kanpur Riemannian Geom. 1999, 95, Agra 76; Banaras Riemannian Geom. 70; Gujrat 70)

Proof. Let A_j^i be mixed tensor. Consider co-ordinate transformations

$$x'^i = x'^i(x^k), \; x''^i = x''^i(x'^k), \text{ that is}$$

$$x^i \longrightarrow x'^i \longrightarrow x''^i$$

$$(i) \longrightarrow (ii) \longrightarrow (iii)$$

$$A_j^i \longrightarrow A_j'^i \longrightarrow A_j''^i$$

In case of $x^i \longrightarrow x'^i$, we have

$$A_\beta'^\alpha = A_q^p \frac{\partial x'^\alpha}{\partial x^p} \frac{\partial x^q}{\partial x'^\beta}. \quad \dots (1)$$

In case of transformation $x'^i \to x''^i$,

$$A_j''^i = A_\beta'^\alpha \frac{\partial x''^i}{\partial x'^\alpha} \frac{\partial x'^\beta}{\partial x''^j} = A_q^p \frac{\partial x'^\alpha}{\partial x^p} \frac{\partial x^q}{\partial x'^\beta} \cdot \frac{\partial x''^i}{\partial x'^\alpha} \frac{\partial x'^\beta}{\partial x''^j}, \text{ by (1)}$$

$$= A_q^p \left(\frac{\partial x'^\alpha}{\partial x^p} \frac{\partial x''^i}{\partial x'^\alpha} \right) \left(\frac{\partial x^q}{\partial x'^\beta} \frac{\partial x'^\beta}{\partial x''^j} \right) = A_q^p \frac{\partial x''^i}{\partial x^p} \frac{\partial x^q}{\partial x''^j}$$

or,

$$A''^i_j = A_q^p \frac{\partial x''^i}{\partial x^p} \frac{\partial x^q}{\partial x''^j}$$

This proves that if we make the direct transformation from (i) to (iii), we get the same law of transformation. This property is expressed as :

Tensor law of transformation possess group property.

2.3. ADDITION AND SUBTRACTION OF TENSORS

Two tensors can be added or subtracted provided they are of the same rank and similar character.

The sum or difference of two tensors is a tensor of the same rank and similar character.

Theorem 3. ***If A_r^{pq} and B_r^{pq} are tensors, then their sum and difference are tensors.*** *(Purvanchal 1997, 95; Roorkee M.E. 1965)*

Or, To prove that the sum of two tensors of the same kind is again a tensor of the same kind.

(Vikram University Ujjain 1995; Kanpur B. Sc. 1993, 1998, 2003, 2004,)

Proof. Let A_r^{pq} and B_r^{pq} be tensors so that they satisfy the tensor law of transformations namely

$$A_k'^{ij} = A_r^{pq} \frac{\partial x'^i}{\partial x^p} \frac{\partial x'^j}{\partial x^q} \frac{\partial x^r}{\partial x'^k} \quad \dots (1)$$

$$B_k'^{ij} = B_r^{pq} \frac{\partial x'^i}{\partial x^p} \frac{\partial x'^j}{\partial x^q} \frac{\partial x^r}{\partial x'^k} \quad \dots (2)$$

(i) The sum of A_r^{pq} and B_r^{pq} is defined as $A_r^{pq} + B_r^{pq} = C_r^{pq}$ (say) ... (3)

To prove that C_r^{pq} is a tensor.

Adding (1) and (2),

$$A_k'^{ij} + B_k'^{ij} = \left(A_r^{pq} + B_r^{pq} \right) \frac{\partial x'^i}{\partial x^p} \frac{\partial x'^j}{\partial x^q} \frac{\partial x^r}{\partial x'^k}.$$

Using (3), we get

$$C_k'^{ij} = C_r^{pq} \frac{\partial x'^i}{\partial x^p} \frac{\partial x'^j}{\partial x^q} \frac{\partial x^r}{\partial x'^k}.$$

This proves that C_r^{pq} satisfies the tensor law of transformation and hence C_r^{pq} is a tensor.

(ii) The difference of A_r^{pq} and B_r^{pq} is defined as

$$A_r^{pq} - B_r^{pq} = D_r^{pq} \text{ (let)} \quad \dots (4)$$

Subtracting (2) from (1) and then using (4)

$$D_k'^{ij} = D_r^{pq} \frac{\partial x'^i}{\partial x^p} \frac{\partial x'^i}{\partial x^q} \frac{\partial x^r}{\partial x'^k}.$$

This confirms the tensor law of transformation.

Hence D_r^{pq} is a tensor, *i.e.*, $A_r^{pq} - B_r^{pq}$ is a tensor.

2.4. MULTIPLICATION OF TENSORS

(Kanpur 1985)

The product of two tensors is a tensor whose rank is the sum of the ranks of the two tensors. More generally, if we multiply a tensor $A^{i_1 i_2 \dots i_l}_{j_1 j_2 \dots j_m}$ (which is co-variant of order m and contravariant of order l) by a tensor $B^{p_1 p_2 \dots p_l'}_{q_1 q_2 \dots q_m'}$ (which is covariant of order m' and contravariant of order l') then the product obtained is

$$A^{i_1 i_2 \dots i_l}_{j_1 j_2 \dots j_m} \quad B^{p_1 p_2 \dots p_l'}_{q_1 q_2 \dots q_m'}$$

a tensor of rank $l + l' + m + m'$. This product tensor is covariant of order $m + m'$ and contravariant of order $l + l'$. Also this product is called **open proudct** or **outer product** of the two tensors. This is proved in the following theorem :

Theorem 4. ***(a) The tensor product of the tensors of the type (r, s) and (r′, s′) is a tensor of the type (r + r′, s + s′).*** *(Kanpur 1986, 84, 83)*

Proof. Let $A^{i_1 i_2 \dots i_r}_{j_1 j_2 \dots j_s}$ be a tensor of the type (r, s).

Let $B^{p_1 p_2 \dots p_r'}_{q_1 q_2 \dots q_s'}$ be a tensor of the type (r', s').

Write $C^{i_1 \dots i_r p_1 \dots p_r'}_{j_1 \dots j_s q_1 \dots q_s'} = A^{i_1 \dots i_r}_{j_1 \dots j_s} \; B^{p_1 \dots p_r'}_{q_1 \dots q_s'}$. ... (1)

This product of tensors is called open product.

Aim. $C^{i_1 \cdots i_r p_1 \cdots p_{r'}}_{j_1 \cdots j_s q_1 \cdots q_s}$ is a tensor of the type $(r + r', s + s')$.

By tensor law of transformations,

$$A'^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s} = A^{i_1 \cdots i_r}_{j_1 \cdots j_s} \frac{\partial x'^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial x'^{\alpha_r}}{\partial x^{i_r}} \cdots \frac{\partial x^{j_1}}{\partial x'^{\beta_1}} \cdots \frac{\partial x^{j_s}}{\partial x'^{\beta_s}} \qquad \ldots (2)$$

$$B'^{a_1 \cdots a_{r'}}_{b_1 \cdots b_{s'}} = B^{p_1 \cdots p_{r'}}_{q_1 \cdots q_{s'}} \frac{\partial x'^{a_1}}{\partial x^{p_1}} \cdots \frac{\partial x'^{a_{r'}}}{\partial x^{p_{r'}}} \frac{\partial x^{q_1}}{\partial x'^{b_1}} \cdots \frac{\partial x^{q_s}}{\partial x'^{b_{s'}}}. \qquad \ldots (3)$$

Multiplying (2) and (3) properly and noting (1),

$$C'^{\alpha_1 \cdots \alpha_r a_1 \cdots a_{r'}}_{\beta_1 \cdots \beta_s b_1 \cdots b_{s'}} = C^{i_1 \cdots i_r p_1 \cdots p_{r'}}_{j_1 \cdots j_s q_1 \cdots q_{s'}} \frac{\partial x'^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial x'^{\alpha_{r'}}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial x'^{\beta_1}} \cdots \frac{\partial x^{j_s}}{\partial x'^{\beta_s}} \frac{\partial x'^{a_1}}{\partial x^{p_1}} \cdots \frac{\partial x'^{a_{r'}}}{\partial x^{p_{r'}}} \frac{\partial x^{q_1}}{\partial x'^{b_1}} \cdots \frac{\partial x^{q_s}}{\partial x'^{b_{s'}}}$$

This confirms that $C^{i_1 \cdots i_r p_1 \cdots p_{r'}}_{j_1 \cdots j_s q_1 \cdots q_{s'}}$ is a tensor of the type $(r + r', s + s')$.

Proved.

Results of outer product of tensors :

(1) The outer product of two contravariant vectors is a contravariant tensor of order 2. Every contravariant tensor of order 2 is not necessarily the tensor (outer) product of two contravariant vectors.

(2) The tensor product of two covariant vectors is a covariant tensor of order 2. But every covariant tensor of order 2 is not necessarily the tensor prouct of two covariant vectors.

(3) The tensor product of a contravariant vector and a covariant vector is a mixed tensor of order 2. But every mixed tensor of order 2 is not necessarily the tensor product of a contravariant vector and a covariant vector.

Theorem 4. ***(b) To prove that the open product of two vectors is a tensor of order two. Is the converse true ?***

(Meerut 1992; Kanpur Riemannian Geom. 1999)

Proof. Let C^i_j be the open product of two vectors A^i and B_j, then

$$C^i_j = A^i B_j. \qquad \ldots (1)$$

Aim. C^i_j is tensor of order 2.

By tensor law of transformation,

$$A'^i = A^\alpha \frac{\partial x'^i}{\partial x^\alpha}, \quad B'_j = B_\beta \frac{\partial x^\beta}{\partial x'^j}$$

Multiplying these properly,

$$A'^i B'_j = A^\alpha B_\beta \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^j}.$$

Using (1), $\quad C'^i_j = C^\alpha_\beta \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^j}.$

This confirms that C^i_j is a tensor of order 2.

Second Part. The converse is not true, *i.e.*, tensor product of two vectors is a tensor of order 2. But every tensor of order 2 is not necessarily the tensor product of two vectors.

Similar Problem. ***Prove that the outer product of two contravariant vectors is a contravariant tensor.***

(Kanpur B. Sc. 1998)

Hint. Write $C^{ij} = A^i B^j$ and proceed as Theorem 4b.

Theorem 4. ***(c) Show that the outer product of two tensors is a tensor whose order is sum of the orders of the two tensors.***

(Kanpur 1984; Banaras Riemannian Geom. 1963)

Proof. Let A^{ij}_k and B^p_{qr} be tensors. Let

$$C^{ijp}_{kqr} = A^{ij}_k B^p_{qr} \qquad \ldots (1)$$

be the outer product of the two tensors.

If we show that C^{ijp}_{kqr} is a tensor of rank 6, the result will follow.

We have
$$A'^{\alpha\beta}_\gamma = A^{ij}_k \frac{\partial x'^\alpha}{\partial x^i} \frac{\partial x'^\beta}{\partial x^j} \frac{\partial x^k}{\partial x'^\gamma} \qquad \ldots (2)$$

$$B'^l_{mt} = B^p_{qr} \frac{\partial x'^l}{\partial x^p} \frac{\partial x^q}{\partial x'^m} \frac{\partial x^r}{\partial x'^t}. \qquad \ldots (3)$$

Multiplying (2) by (3) properly and noting (1), we get

$$A'^{\alpha\beta}_\gamma B'^l_{mt} = C^{ijp}_{kqr} \frac{\partial x'^\alpha}{\partial x^i} \frac{\partial x'^\beta}{\partial x^j} \frac{\partial x^k}{\partial x'^\gamma} \frac{\partial x'^l}{\partial x^p} \frac{\partial x^q}{\partial x'^m} \frac{\partial x^r}{\partial x'^t}.$$

By virtue of (1), this gives

$$C'^{\alpha\beta l}_{\gamma mt} = C^{ijp}_{kqr} \frac{\partial x'^\alpha}{\partial x^i} \frac{\partial x'^\beta}{\partial x^j} \frac{\partial x'^l}{\partial x^p} \frac{\partial x^k}{\partial x'^\gamma} \frac{\partial x^q}{\partial x'^m} \frac{\partial x^r}{\partial x'^t}.$$

This confirms the law of transformation of a mixed tensor of rank 6. Hence C^{ijp}_{kqr}, *i.e.*, $A^{ij}_k B^p_{qr}$ is a mixed tensor of rank 6.

(Kanpur 1985, 87)

2.5. INNER PRODUCT

Let A^α be a contravariant vector, and B_α covariant vector. The product $A^\alpha B_\alpha$ is called *scalar* or *inner product* of the vectors A^α and B_α. This scalar product is an invariant, *i.e.*, it has the same value in any set of co-ordinates.

For
$$A'^i B'_i = A^\alpha \frac{\partial x'^i}{\partial x^\alpha} B_p \frac{\partial x^p}{\partial x'^i}$$
$$= A^\alpha B_p \frac{\partial x'^i}{\partial x^\alpha} \frac{\partial x^p}{\partial x'^i}$$
$$= A^\alpha B_p \frac{\partial x^p}{\partial x^\alpha} = A^\alpha B_p \delta^p_\alpha = A^\alpha B_\alpha = A^i B_i$$

i.e., $\quad A'^i B'_i = A^i B_i.$

This proves that :

"The inner product of covariant and contravariant vectors is a scalar invariant." ***(Gorakhpur 1995)***

The product $A^{ij} B_{jp}$ of two tensors A^{ij} and B_{jp} is called an *inner product* of the two tensors. This process is called **inner multiplication** of the two tensors. The inner product is a tensor.

That is to say, if we set in a product of two tensors one contravariant and one covariant suffixes equal, the process is called inner multiplication and the resulting tensor is called the inner product of the two tensors. For example,

$$A^{ij}_k B^k_{pqr}, A^{ij}_k B^s_{iqr}, A^{ij}_k B^s_{pjr}$$

all the inner products of the tensors

$$A_k^{ij} \text{ and } B_{pqr}^s.$$

Theorem 5. ***To show that the inner product of the tensor A_r^p and B_t^{qs} is a tensor of rank three.***

(Kanpur B. Sc. 1995; M. Sc. 98, 97; Roorkee M.E. 1967)

Proof. The inner product of tensors A_r^p and B_t^{qs} is $A_r^p B_t^{rs}$. We want to show that $A_r^p B_t^{rs}$ is a tensor of rank three.

By tensor law of transformation,

$$A'^p_r B'^{rs}_t = A^\alpha_\beta \frac{\partial x'^p}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^r} \cdot B^{lm}_q \frac{\partial x'^r}{\partial x^l} \frac{\partial x'^s}{\partial x^m} \frac{\partial x^q}{\partial x'^t}.$$

$$= A^\alpha_\beta B^{lm}_q \frac{\partial x'^p}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^r} \frac{\partial x'^r}{\partial x^l} \frac{\partial x'^s}{\partial x^m} \frac{\partial x^q}{\partial x'^t}$$

$$= A^\alpha_\beta B^{lm}_q \frac{\partial x'^p}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x^l} \frac{\partial x'^s}{\partial x^m} \frac{\partial x^q}{\partial x'^t}$$

$$= A^\alpha_\beta B^{lm}_q \frac{\partial x'^p}{\partial x^\alpha} \delta^\beta_l \frac{\partial x'^s}{\partial x^m} \frac{\partial x^q}{\partial x'^t}$$

$$= A^\alpha_l B^{lm}_q \frac{\partial x'^p}{\partial x^\alpha} \frac{\partial x'^s}{\partial x^m} \frac{\partial x^q}{\partial x'^t}$$

$$A'^p_r B'^{rs}_t = A^\alpha_l B^{lm}_q \frac{\partial x'^p}{\partial x^\alpha} \frac{\partial x'^s}{\partial x^m} \frac{\partial x^q}{\partial x'^t}.$$

But this is the law of transformation of a mixed tensor of rank 3. Hence $A_r^p B_t^{rs}$ is a mixed tensor of rank 3.